

# Estimations of the discrete Green's function of the SDFEM on Shishkin triangular meshes for problems with only exponential layers

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## Abstract

In this technical report, we present estimations of the discrete Green's function of the streamline diffusion finite element method (SDFEM) on Shishkin triangular meshes for problems with only exponential layers.

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## 1. Continuous problem, Shishkin mesh, SDFEM

We consider the singularly perturbed boundary value problem

$$\begin{aligned} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + cu &= f & \text{in } \Omega = (0, 1)^2, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where  $\varepsilon \ll |\mathbf{b}|$  is a small positive parameter,  $\mathbf{b} = (b_1, b_2)^T$  is a constant vector with  $b_1 > 0, b_2 > 0$  and  $c > 0$  is constant. It is also assumed that  $f$  is sufficiently smooth. The solution of (1) typically has two exponential layers of width  $O(\varepsilon \ln(1/\varepsilon))$  at the sides  $x = 1$  and  $y = 1$  of  $\Omega$ .

When discretizing (1), we use *Shishkin* meshes, which are piecewise uniform. See [3, 4, 1] for a detailed discussion of their properties and applications.

First, we define two mesh transition parameters, which are used to specify mesh changes from coarse to fine in  $x$ - and  $y$ -direction,

$$\lambda_x := \min \left\{ \frac{1}{2}, \rho \frac{\varepsilon}{\beta_1} \ln N \right\} \quad \text{and} \quad \lambda_y := \min \left\{ \frac{1}{2}, \rho \frac{\varepsilon}{\beta_2} \ln N \right\}.$$

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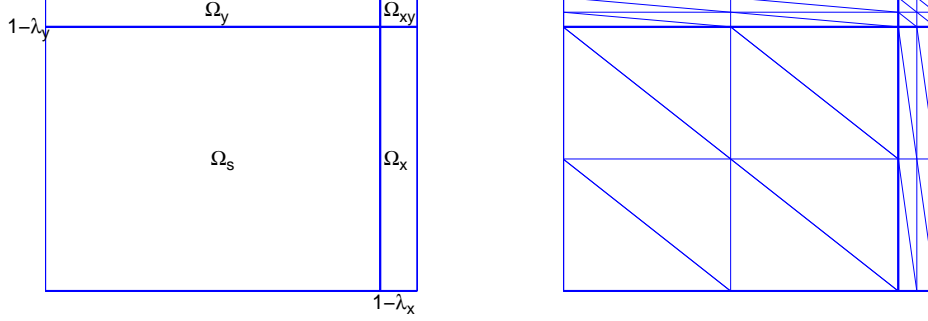


Figure 1: Dissection of  $\Omega$  and triangulation  $\mathcal{T}_N$ .

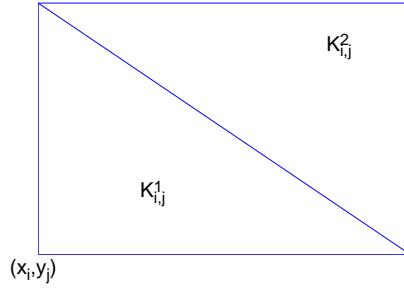


Figure 2:  $K_{i,j}^1$  and  $K_{i,j}^2$

For technical reasons, we set  $\rho = 2.5$  in our analysis which is the same with ones in [6] and [5]. We divide  $\Omega$  as in Fig. 1:  $\overline{\Omega} = \Omega_s \cup \Omega_x \cup \Omega_y \cup \Omega_{xy}$ , where

$$\begin{aligned} \Omega_s &:= [0, 1 - \lambda_x] \times [0, 1 - \lambda_y], & \Omega_x &:= [1 - \lambda_x, 1] \times [0, 1 - \lambda_y], \\ \Omega_y &:= [0, 1 - \lambda_x] \times [1 - \lambda_y, 1], & \Omega_{xy} &:= [1 - \lambda_x, 1] \times [1 - \lambda_y, 1]. \end{aligned}$$

**Assumption 1.** *We assume that  $\varepsilon \leq N^{-1}$ , as is generally the case in practice. Furthermore we assume that  $\lambda_x = \rho\varepsilon\beta_1^{-1} \ln N$  and  $\lambda_y = \rho\varepsilon\beta_2^{-1} \ln N$  as otherwise  $N^{-1}$  is exponentially small compared with  $\varepsilon$ .*

Next, we define the set of mesh points  $\{(x_i, y_j) \in \Omega : i, j = 0, \dots, N\}$

$$x_i = \begin{cases} 2i(1 - \lambda_x)/N & \text{for } i = 0, \dots, N/2, \\ 1 - 2(N - i)\lambda_x/N & \text{for } i = N/2 + 1, \dots, N \end{cases}$$

and

$$y_j = \begin{cases} 2j(1 - \lambda_y)/N & \text{for } j = 0, \dots, N/2, \\ 1 - 2(N - j)\lambda_y/N & \text{for } j = N/2 + 1, \dots, N. \end{cases}$$

By drawing lines through these mesh points parallel to the  $x$ -axis and  $y$ -axis the domain  $\Omega$  is partitioned into rectangles. Each rectangle is divided into two triangles by drawing the diagonal. This yields a triangulation of  $\Omega$  denoted by  $\mathcal{T}_N$  (see Fig. 1). The mesh sizes  $h_{x,i} := x_{i+1} - x_i$  and  $h_{y,j} := y_{j+1} - y_j$  satisfy

$$h_{x,i} = \begin{cases} H_x := \frac{1 - \lambda_x}{N/2} & \text{for } i = 0, \dots, N/2 - 1, \\ h_x := \frac{\lambda_x}{N/2} & \text{for } i = N/2, \dots, N - 1 \end{cases}$$

and

$$h_{y,j} = \begin{cases} H_y := \frac{1 - \lambda_y}{N/2} & \text{for } j = 0, \dots, N/2 - 1, \\ h_y := \frac{\lambda_y}{N/2} & \text{for } j = N/2, \dots, N - 1. \end{cases}$$

The mesh sizes  $H_x, H_y, h_x$  and  $h_y$  satisfy

$$N^{-1} \leq H_x, H_y \leq 2N^{-1} \quad \text{and} \quad C_1 \varepsilon N^{-1} \ln N \leq h_x, h_y \leq C_2 \varepsilon N^{-1} \ln N.$$

For convenience, we shall use the following notations:  $K_{i,j}^1$  for the mesh triangle with vertices  $(x_i, y_j)$ ,  $(x_{i+1}, y_j)$ , and  $(x_i, y_{j+1})$ ;  $K_{i,j}^2$  for the mesh triangle with vertices  $(x_i, y_{j+1})$ ,  $(x_{i+1}, y_j)$ , and  $(x_{i+1}, y_{j+1})$  (see Fig. 2);  $K$  for a generic mesh triangle.

On the above Shishkin meshes we define a  $C^0$  linear finite element space

$$V^N := \{v^N \in C(\bar{\Omega}) : v^N|_{\partial\Omega} = 0 \text{ and } v^N|_K \in P_1(K), \forall K \in \mathcal{T}_N\}.$$

Now we are in a position to state the SDFEM. Let  $V := H_0^1(\Omega)$  and define the bilinear forms

$$\begin{aligned} a_{Gal}(v, w) &= \varepsilon(\nabla v, \nabla w) + (\mathbf{b} \cdot \nabla v, w) + (cv, w) \quad v, w \in V; \\ a_{stab}(v, w) &= \sum_{K \subset \Omega} (-\varepsilon \Delta v + \mathbf{b} \cdot \nabla v + cv, \delta_K \mathbf{b} \cdot \nabla w)_K \quad v \in V \cap H^2(\mathcal{T}_N), w \in V; \\ a_{SD}(v, w) &= a_{Gal}(v, w) + a_{stab}(v, w) \quad v \in V \cap H^2(\mathcal{T}_N), w \in V, \end{aligned}$$

where  $H^2(\mathcal{T}_N) = \{v \in L^2(\Omega) : \forall K \in \mathcal{T}_N, v|_K \in H^2(K)\}$ . The standard SDFEM reads:

$$\begin{cases} \text{Find } u^N \in V^N \text{ such that for all } v^N \in V^N, \\ a_{SD}(u^N, v^N) = (f, v^N) + \sum_{K \subset \Omega} (f, \delta_K \mathbf{b} \cdot \nabla v^N)_K. \end{cases} \quad (2)$$

Note that  $\Delta u^N = 0$  in  $K$  for  $u^N|_K \in P_1(K)$ . Following usual practice [4], the parameter  $\delta_K := \delta|_K$  is defined as follows:

$$\delta_K = \begin{cases} C^* N^{-1} & \text{if } K \subset \Omega_s, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

and  $C^*$  is a properly defined positive constant such that the following coercivity holds (see [4, Lemma 3.25]):

$$a_{SD}(v^N, v^N) \geq \frac{1}{2} \|v^N\|^2 \quad \forall v^N \in V^N, \quad (4)$$

where

$$\|v^N\|^2 := \varepsilon |v^N|_1^2 + \|v^N\|^2 + \sum_{K \subset \Omega} \delta_K \|\mathbf{b} \cdot \nabla v^N\|_K^2. \quad (5)$$

Coercivity (4) implies a unique solution of the discrete problem (2). Also the Galerkin orthogonality holds, i.e.,

$$a_{SD}(u - u^N, v^N) = 0 \quad \forall v^N \in V^N. \quad (6)$$

Set

$$b := \sqrt{b_1^2 + b_2^2}, \quad \boldsymbol{\beta} := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} / b, \quad \boldsymbol{\eta} := \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} / b.$$

For our later analysis, we define a mesh subdomain of  $\Omega$  for each mesh node  $\mathbf{x}^* = (x^*, y^*)$ :

$$\Omega'_0 := \Omega'_0(\mathbf{x}^*) = \{K \in \mathcal{T}_N : \text{meas}(\Omega_0 \cap K) \neq 0\}, \quad (7)$$

where

$$\Omega_0 := \Omega_0(\mathbf{x}^*) = \left\{ \mathbf{x} = (x, y) \in \Omega : (\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\beta} \leq \mathcal{K} \sigma_\beta \ln N \text{ and } |(\mathbf{x} - \mathbf{x}^*) \cdot \boldsymbol{\eta}| \leq \mathcal{K} \sigma_\eta \ln N \right\} \quad (8)$$

(see Fig. 3 and Fig. 4) and

$$\sigma_\beta = k N^{-1} \ln N, \quad \sigma_\eta = k N^{-1/2}. \quad (9)$$

We shall choose  $k > 0$  and  $\mathcal{K} > 0$  later, which are independent of  $N$  and  $\varepsilon$ . Note that

$$\text{meas}(\Omega'_0) \leq C \sigma_\eta \ln N. \quad (10)$$

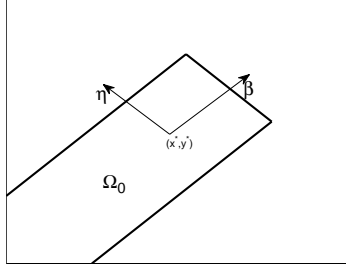


Figure 3: Subdomain  $\Omega_0 = \Omega_0(\mathbf{x}^*)$

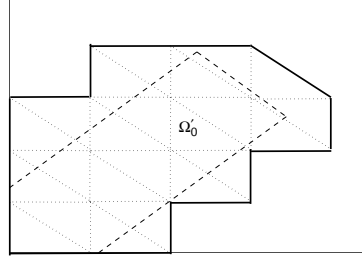


Figure 4: Subdomain  $\Omega'_0 = \Omega'_0(\mathbf{x}^*)$

## 2. The discrete Green's function

In this section, we introduce the discrete Green's function and cite some results from [2].

Let  $\mathbf{x}^* = (x^*, y^*)$  be a mesh node in  $\Omega$ . The discrete Green's function  $G \in V^N$  associated with  $\mathbf{x}^*$  is defined by

$$a_{SD}(v^N, G) = v^N(\mathbf{x}^*) \quad \forall v^N \in V^N. \quad (11)$$

In the following analysis, we present the energy estimation of discrete Green's functions. Our analysis is similar to that of [2], with some changes.

We define a weight function

$$\omega(\mathbf{x}) := g\left(\frac{(\mathbf{x} - \mathbf{x}^*) \cdot \beta}{\sigma_\beta}\right) g\left(\frac{(\mathbf{x} - \mathbf{x}^*) \cdot \eta}{\sigma_\eta}\right) g\left(-\frac{(\mathbf{x} - \mathbf{x}^*) \cdot \eta}{\sigma_\eta}\right)$$

with  $g(r) = 2/(1 + e^r)$  for  $r \in (-\infty, \infty)$ . We shall choose  $k > 0$  later. Note Lemma 4.1 in [2] holds if  $\sigma_\beta \geq kN^{-1} \ln N$  and  $\sigma_\eta \geq kN^{-1/2}$ .

Now we define a weighted energy norm

$$\begin{aligned} \|G\|_\omega^2 := & \varepsilon \|\omega^{-1/2} G_\beta\|^2 + \varepsilon \|\omega^{-1/2} G_\eta\|^2 + \frac{b}{2} \|(\omega^{-1})_\beta^{1/2} G\|^2 \\ & + c \|\omega^{-1/2} G\|^2 + \sum_K b^2 \delta_K \|\omega^{-1/2} G_\beta\|_K^2. \end{aligned} \quad (12)$$

Note that  $(\omega^{-1})_\beta > 0$ . For any subdomain  $D$  of  $\Omega$ , let  $\|G\|_{\omega, D}$  mean that the integrations in (12) are restricted to  $D$ . From (2), (12) and integration

by parts, we have

$$\begin{aligned} \|G\|_\omega^2 &= a_{SD}(\omega^{-1}G, G) - \varepsilon((\omega^{-1})_\beta G, G_\beta) - \varepsilon((\omega^{-1})_\eta G, G_\eta) \\ &\quad - \sum_K (b(\omega^{-1})_\beta G + c\omega^{-1}G, \delta_K bG_\beta)_K. \end{aligned}$$

Considering (11), we have

$$\begin{aligned} a_{SD}(\omega^{-1}G, G) &= a_{SD}(\omega^{-1}G - (\omega^{-1}G)^I, G) + a_{SD}((\omega^{-1}G)^I, G) \\ &= a_{SD}(\omega^{-1}G - (\omega^{-1}G)^I, G) + (\omega^{-1}G)(\mathbf{x}^*). \end{aligned}$$

By means of the above two equalities, the energy estimate of  $G$  will be obtained from the next three Lemmas.

**Lemma 1.** *Assume  $\sigma_\beta \geq kN^{-1}$  and  $\sigma_\eta \geq k\varepsilon^{1/2}$  in (8), then for  $k > 1$  sufficiently large and independent of  $N$  and  $\varepsilon$ , we have*

$$a_{SD}(\omega^{-1}G, G) \geq \frac{1}{4} \|G\|_\omega^2.$$

*Proof.* See [2, Lemma 4.2]. □

**Lemma 2.** *If  $\sigma_\beta \geq kN^{-1}$  in (8), with  $k > 0$  independent of  $N$  and  $\varepsilon$ , then for each mesh point  $\mathbf{x}^* \in \Omega \setminus \Omega_{xy}$ , we have*

$$|(\omega^{-1}G)(\mathbf{x}^*)| \leq \frac{1}{16} \|G\|_\omega^2 + \begin{cases} CN^2\sigma_\beta & \text{if } \mathbf{x}^* \in \Omega_s \\ CN \ln N & \text{if } \mathbf{x}^* \in \Omega_x \cup \Omega_y \end{cases}.$$

where  $C$  is independent of  $N$ ,  $\varepsilon$  and  $\mathbf{x}^*$ .

*Proof.* See [2, Lemma 4.3]. □

**Lemma 3.** *If  $\sigma_\beta \geq kN^{-1} \ln N$  and  $\sigma_\eta \geq kN^{-1/2}$  in (8), where  $k > 1$  is sufficiently large and independent of  $N$  and  $\varepsilon$ , then*

$$a_{SD}((\omega^{-1}G)^I - \omega^{-1}G, G) \leq \frac{1}{16} \|G\|_\omega^2.$$

*Proof.* For convenience we set  $\tilde{E}(\mathbf{x}) := ((\omega^{-1}G)^I - \omega^{-1}G)(\mathbf{x})$ . Recall  $b$  is constant and integration by parts yields  $(b\tilde{E}_\beta, G) = -(b\tilde{E}, G_\beta)$ . Then we have

$$\begin{aligned} |a_{SD}(\tilde{E}, G)| &\leq C(\|(\varepsilon + b^2\delta)^{1/2}\omega^{1/2}\tilde{E}_\beta\| + \varepsilon^{1/2}\|\omega^{1/2}\tilde{E}_\eta\| \\ &\quad + \|(\varepsilon + b^2\delta)^{-1/2}\omega^{1/2}\tilde{E}\|) \|G\|_\omega. \end{aligned} \tag{13}$$

To analyze different kinds of interpolation bounds, we first estimate the following terms. Note that  $G_{\beta\beta} = G_{\eta\eta} = G_{\beta\eta} = 0$  on  $K$  because  $G$  belongs to  $V^N$ . For convenience, we set  $M_K := \max_K \omega^{-1/2}$ . Using (iii), (iv) and (v) in [2, Lemma 4.1], we obtain

$$\begin{aligned} \|(\omega^{-1}G)_{\beta\beta}\|_K &\leq \|(\omega^{-1})_{\beta\beta}G\|_K + \|(\omega^{-1})_{\beta}G_{\beta}\|_K \\ &\leq CM_K \left( \sigma_{\beta}^{-3/2} \|(\omega^{-1})_{\beta}^{1/2}G\|_K + \sigma_{\beta}^{-1} \|\omega^{-1/2}G_{\beta}\|_K \right) \\ &\leq CM_K \left( \sigma_{\beta}^{-3/2} + \sigma_{\beta}^{-1}(\varepsilon + b^2\delta)^{-1/2} \right) \|G\|_{\omega,K}. \end{aligned} \quad (14)$$

Note  $\|G_{\eta}\|_K \leq C \max\{h_{x,K}^{-1}, h_{y,K}^{-1}\} \|G\|_K$  or  $\|G_{\eta}\|_K \leq C\varepsilon^{-1/2} \cdot \varepsilon^{1/2} \|G_{\eta}\|_K$ , then we have

$$\|G_{\eta}\|_K \leq C \min\{\max\{h_{x,K}^{-1}, h_{y,K}^{-1}\}, \varepsilon^{-1/2}\} \|G\|_K,$$

and

$$\begin{aligned} \|(\omega^{-1})_{\eta}G_{\eta}\|_K &\leq C \max_K |(\omega^{-1})_{\eta}| \|G_{\eta}\|_K \\ &\leq CM_K \sigma_{\eta}^{-1} \max_K \omega^{-1/2} \cdot \min\{\max\{h_{x,K}^{-1}, h_{y,K}^{-1}\}, \varepsilon^{-1/2}\} \|G\|_K \\ &\leq CM_K \sigma_{\eta}^{-1} \min\{\max\{h_{x,K}^{-1}, h_{y,K}^{-1}\}, \varepsilon^{-1/2}\} \|G\|_{\omega,K}. \end{aligned} \quad (15)$$

Similarly, we have  $\|(\omega^{-1})_{\eta\eta}G\|_K \leq CM_K \sigma_{\eta}^{-2} \|\omega^{-1/2}G\|_K$  and

$$\begin{aligned} \|(\omega^{-1}G)_{\eta\eta}\|_K &\leq \|(\omega^{-1})_{\eta\eta}G\|_K + \|(\omega^{-1})_{\eta}G_{\eta}\|_K \\ &\leq CM_K \left( \sigma_{\eta}^{-2} + \sigma_{\eta}^{-1} \min\{\max\{h_{x,K}^{-1}, h_{y,K}^{-1}\}, \varepsilon^{-1/2}\} \right) \|G\|_{\omega,K}. \end{aligned} \quad (16)$$

Recalling (v) in [2, Lemma 4.1] and inverse estimates, we have

$$\begin{aligned} \|(\omega^{-1})_{\beta}G_{\eta}\|_K &\leq C \max_K (\omega^{-1})_{\beta} \cdot \|G_{\eta}\|_K \\ &\leq C \max_K (\omega^{-1})_{\beta} \cdot \max\{h_{x,K}^{-1}, h_{y,K}^{-1}\} \cdot \|G\|_K \\ &\leq C \max\{h_{x,K}^{-1}, h_{y,K}^{-1}\} \left( \max_K (\omega^{-1})_{\beta} \right)^{1/2} \left( \min_K (\omega^{-1})_{\beta} \right)^{1/2} \cdot \|G\|_K \\ &\leq CM_K \cdot \max\{h_{x,K}^{-1}, h_{y,K}^{-1}\} \sigma_{\beta}^{-1/2} \cdot \|(\omega^{-1})_{\beta}^{1/2}G\|_K. \end{aligned} \quad (17)$$

Also, we have

$$\|(\omega^{-1})_{\beta}G_{\eta}\|_K \leq CM_K \cdot \varepsilon^{-1/2} \sigma_{\beta}^{-1} \cdot \varepsilon^{1/2} \|\omega^{-1/2}G_{\eta}\|_K. \quad (18)$$

Then from (17) and (18), we have

$$\|(\omega^{-1})_\beta G_\eta\|_K \leq CM_K \min\{\max\{h_{x,K}^{-1}, h_{y,K}^{-1}\}\sigma_\beta^{-1/2}, \varepsilon^{-1/2}\sigma_\beta^{-1}\}\|G\|_{\omega,K},$$

and then

$$\begin{aligned} & \|(\omega^{-1}G)_{\beta\eta}\|_K \leq \|(\omega^{-1})_{\beta\eta}G\|_K + \|(\omega^{-1})_\eta G_\beta\|_K + \|(\omega^{-1})_\beta G_\eta\|_K \\ & \leq CM_K \left( \sigma_\beta^{-1/2}\sigma_\eta^{-1}\|(\omega^{-1})_\beta^{1/2}G\|_K + \sigma_\eta^{-1}\|\omega^{-1/2}G_\beta\|_K \right) + \|(\omega^{-1})_\beta G_\eta\|_K \\ & \leq CM_K (\sigma_\beta^{-1/2}\sigma_\eta^{-1} + \sigma_\eta^{-1}(\varepsilon + b^2\delta)^{-1/2} \\ & \quad + \min\{\max\{h_{x,K}^{-1}, h_{y,K}^{-1}\}\sigma_\beta^{-1/2}, \varepsilon^{-1/2}\sigma_\beta^{-1}\})\|G\|_{\omega,K}. \end{aligned} \quad (19)$$

Set  $h_K = \max\{h_{x,K}, h_{y,K}\}$  and  $\tilde{M}_K := (\max_K \omega)^{1/2}$ . From [2, Corollary 3.1], we have

$$\begin{aligned} & \|\omega^{1/2}\tilde{E}\|_K + h_K\|\omega^{1/2}\tilde{E}_\beta\|_K + h_K\|\omega^{1/2}\tilde{E}_\eta\|_K \\ & \leq C\tilde{M}_K h_K^2 (\|(\omega^{-1}G)_{\beta\beta}\|_K + \|(\omega^{-1}G)_{\beta\eta}\|_K + \|(\omega^{-1}G)_{\eta\eta}\|_K). \end{aligned} \quad (20)$$

Substituting (14), (16) and (19) into (20), we have

$$(\varepsilon + b^2\delta)\|\omega^{1/2}\tilde{E}_\beta\|^2 + \varepsilon\|\omega^{1/2}\tilde{E}_\eta\|^2 \leq Ck^{-2}\|G\|_\omega^2. \quad (21)$$

More precisely, we have

$$\begin{aligned} \|\omega^{1/2}\tilde{E}_\beta\|_{\Omega_x} & \leq Ck^{-1}N^{-1}(\sigma_\beta^{-1}\varepsilon^{-1/2} + \sigma_\eta^{-2} + \sigma_\eta^{-1}\varepsilon^{-1/2})\|G\|_\omega \\ & \leq Ck^{-1}\varepsilon^{-1/2}\ln^{-1}N\|G\|_\omega. \end{aligned} \quad (22)$$

Substituting (14), (16) and (19) into (20) again, we have

$$\|\omega^{1/2}\tilde{E}\|_K \leq \begin{cases} Ck^{-1}N^{-1/2}\|G\|_{\omega,K} & \text{if } K \subset \Omega_s \\ Ck^{-1}\varepsilon^{1/2}\|G\|_{\omega,K} & \text{if } K \subset \Omega_{xy} \end{cases}, \quad (23)$$

and

$$\|\omega^{1/2}\tilde{E}\|_K \leq Ck^{-1}\varepsilon^{-1/2}N^{-1}\ln^{-1}N\|G\|_{\omega,K} \text{ if } K \subset \Omega_x \cup \Omega_y. \quad (24)$$

For what follows we need a sharper bound of  $\|\omega^{1/2}\tilde{E}\|_{\Omega_x \cup \Omega_y}$ . Considering  $\sigma_\beta \geq kN^{-1}\ln N$ ,  $\sigma_\eta \geq kN^{-1/2}$ , (24) and (22), similar to [2, Lemma 4.4] we obtain

$$\begin{aligned} & \|\omega^{1/2}\tilde{E}\|_{\Omega_x \cup \Omega_y}^2 \leq C\lambda_x^2 \left\{ \|(\omega^{1/2})_\beta \tilde{E}\|_{\Omega_x \cup \Omega_y}^2 + \|\omega^{1/2}\tilde{E}_\beta\|_{\Omega_x \cup \Omega_y}^2 \right\} \\ & \leq C\varepsilon^2 \ln^2 N \cdot \left\{ \sigma_\beta^{-2}\|\omega^{1/2}\tilde{E}\|_{\Omega_x \cup \Omega_y}^2 + \|\omega^{1/2}\tilde{E}_\beta\|_{\Omega_x \cup \Omega_y}^2 \right\} \\ & \leq Ck^{-2}\varepsilon^2 \ln^2 N \{ \sigma_\beta^{-2}\varepsilon^{-1}N^{-2}\ln^{-2}N + \varepsilon^{-1}\ln^{-2}N \} \|G\|_\omega^2 \\ & \leq Ck^{-2}\varepsilon\|G\|_\omega^2. \end{aligned} \quad (25)$$



Substituting (21), (23) and (25) into (13) and recalling the definition of  $\delta$ , we obtain

$$|a_{SD}(\tilde{E}, G)| \leq Ck^{-1} \|G\|_{\omega}^2.$$

Choosing  $k$  sufficiently large independently of  $\varepsilon$  and  $N$ , we are done.  $\square$

Lemmas 1, 2 and 3 yield the following bound of the discrete Green function in the energy norm just as in [2, Theorem 4.1].

**Theorem 1.** *Assume that  $\sigma_{\beta} = kN^{-1} \ln N$  and  $\sigma_{\eta} = kN^{-1/2}$  in (8), where  $k$  is chosen so that Lemmas 1, 2 and 3 hold. Then for  $\mathbf{x}^* \in \Omega \setminus \Omega_{xy}$ , we have*

$$\|G\| \leq \sqrt{8} \|G\|_{\omega} \leq CN^{1/2} \ln^{1/2} N.$$

**Remark 1.** *For different crosswind diffusion coefficients, for example  $\hat{\varepsilon}$  in [2] and  $\varepsilon$  in the present paper, we can choose  $\sigma_{\beta}$  and  $\sigma_{\eta}$  large enough such that Lemmas 1–3 hold true. Thus with different assumptions on these parameters and different SDFEMs, we can obtain bounds similar to [2, Theorem 4.1], i.e., Theorem 1.*

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